5-manifolds admitting rank two distributions of Cartan type
(based on joint work with Shantanu Dave)

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Outline of the talk

1. A geometry in five dimensions
2. Global existence
3. Analysis of BGG sequences
A geometry in five dimensions

Definition (distributions of Cartan type on 5-manifolds)

A rank two distribution $\xi \subseteq TM$ is called of Cartan type if it is bracket generating with growth vector $(2,3,5)$, i.e., if locally there exist sections $X, Y \in \Gamma^\infty(\xi)$ such that $X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]]$ is a frame of $TM$. 

a.k.a. generic rank two distributions in dimension five

Lie group $\mathcal{N}$ with graded nilpotent Lie algebra $n = n_1 \oplus n_2 \oplus n_3$:

$X, Y \in n_1$, $[X, Y] \in n_2$, $[X, [X, Y]], [Y, [X, Y]] \in n_3$ 

$G_2/P \sim = S^2 \times S^3$ (flat model) 

surface rolling on another surface (5-dim. configuration space)

without slipping and twisting (encoded in rank two distribution) 

is of Cartan type iff Gaussian curvatures disjoint
A geometry in five dimensions

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- Lie group $N$ with graded nilpotent Lie algebra $n = n_1 \oplus n_2 \oplus n_3$:

  $$X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]]$$

  $n_1 \quad n_2 \quad n_3$

- $G_2/P \cong S^2 \times S^3$ (flat model)

- surface rolling on another surface (5-dim. configuration space) without slipping and twisting (encoded in rank two distribution) is of Cartan type iff Gaussian curvatures disjoint
A geometry in five dimensions (cont.)

**Parabolic geometry** of type \((G_2, P)\):

- canonical Cartan connection
- symmetry group of dimension at most 14
- Cartan’s curvature tensor (a section of \(S^4\xi^*\)) vanishes iff locally diffeomorphic to flat model \(G_2/P\)
- curved BGG sequences [Čap–Slovák–Souček]
- canonical conformal metric of signature \((2, 3)\) with conformal holonomy contained in \(G_2\) [Nurowski]
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Filtered manifold: $0 \subseteq \xi \subseteq \eta \subseteq TM$ of type $(2, 3, 5)$
induced (tensorial) Levi bracket on

$$\text{gr}(TM) = \xi \oplus (\eta/\xi) \oplus (TM/\eta)$$

osculating algebras $\text{gr}(T_xM) \cong n = n_1 \oplus n_2 \oplus n_3$
Existence of global structure

Basic question
Which 5-manifolds admit rank two distribution of Cartan type?
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Results for contact and Engel structures:
- On open manifolds the inclusion
  \[ \{\text{contact structures}\} \subseteq \{\text{almost contact structures}\} \]
  is a (weak) homotopy equivalence [Gromov’s h-principle]
- Closed oriented 3-manifolds admit contact structures [Lutz-Martinez]
- Dichotomy on closed 3-manifolds: tight — overtwisted
  For a fixed embedded disc \(D^2\) the inclusion
  \[ \{\text{overtwisted contact structures with overtwisted disc } D^2\} \subseteq \{\text{almost contact structures with overtwisted disc } D^2\} \]
  is a (weak) homotopy equivalence [Eliashberg]
- Similar picture in all odd dimensions [Borman–Eliashberg–Murphy]
- Every parallelizable 4-manifold admits Engel structure [Vogel]
  i.e. rank two distribution with growth vector \((2, 3, 4)\)
If $M$ admits orientable distribution of Cartan type $\xi \subseteq TM$, then $TM \cong \xi \oplus \varepsilon_1 \oplus \xi$. (1)

In particular, $M$ spinnable and $\frac{1}{2}p_1(M) = e(\xi)^2$.

If $M$ closed, then $\int M = 0 = k(M) = \sum_{q \text{ even}} \dim H^q(M; \mathbb{R}) \mod 2$.

Theorem (Dave and H.)

a) If $M$ open, spinnable, and $\frac{1}{2}p_1(M) = e(\xi)^2$, then (1) holds.

b) If $M$ closed, spinnable, $\frac{1}{2}p_1(M) = e(\xi)^2$, and $k(M) = 0$, then (1) holds.

Proof.

a) $\frac{1}{2}p_1: \text{BSpin}(5) \to K(\mathbb{Z}, 4)$ is 5-equivalence.

5-manifolds admitting distribution of Cartan type

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If $M$ is closed, then [Atiyah]

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If $M$ admits orientable distribution of Cartan type $\xi \subseteq TM$, then
\[ TM \cong \xi \oplus \varepsilon^1 \oplus \xi. \tag{1} \]

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\textbf{Theorem (Dave and H.)}

\begin{itemize}
  \item[a)] If $M$ open, spinnable, and $\frac{1}{2} p_1(M) = e(\xi)^2$, then (1) holds.
  \item[b)] If $M$ closed, spinnable, $\frac{1}{2} p_1(M) = e(\xi)^2$, and $k(M) = 0$, then (1) holds.
\end{itemize}

\textbf{Proof.}

\begin{itemize}
  \item[a)] $\frac{1}{2} p_1 : B\text{Spin}(5) \to K(\mathbb{Z}, 4)$ is 5-equivalence
  \item[b)] builds on work of Thomas, Atiyah–Dupont, and Tang–Zhang
\end{itemize}
Distributions of Cartan type on open 5-manifolds

Suppose \( M \) is an open, spinnable 5-manifold, \( e \in H^2(M; \mathbb{Z}) \) such that \( e^2 = \frac{1}{2} p_1(M) \).

Then there exists an orientable \( \xi \subseteq TM \) of Cartan type with \( e(\xi) = e \).

Proof. According to Gromov's h-principle.

\[ \Gamma(J^2 Gr^2(TM) \to M) \cong \Gamma(R \to M) \cong \Gamma(F/H \to M) \]

where \( L: \mathbb{R} \to F/H \), \( H = \{ (A^{000} \det A^{000} 0 \det A) : A \in GL_2(\mathbb{R}) \} \subseteq GL_5(\mathbb{R}) \).
Distributions of Cartan type on open 5-manifolds

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Proof.

$$\Gamma(\text{Gr}_2(TM) \to M) \xleftarrow{\mathcal{C}}$$
Distributions of Cartan type on open 5-manifolds

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**Proof.**

\[
\begin{array}{c}
\Gamma (\text{Gr}_2(TM) \to M) \leftarrow \mathcal{C} \\
\downarrow j^2 \\
\Gamma (J^2 \text{Gr}_2(TM) \to M)
\end{array}
\]
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Proof.

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\begin{array}{c}
\Gamma(\text{Gr}_2(TM) \to M) \leftarrow C \\
\downarrow j^2 \quad \quad \quad \quad \quad \downarrow j^2 \\
\Gamma(J^2 \text{Gr}_2(TM) \to M) \leftarrow \Gamma(\mathcal{R} \to M)
\end{array}
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Distributions of Cartan type on open 5-manifolds

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Suppose $M$ open, spinnable 5-manifold, $e \in H^2(M; \mathbb{Z})$ s.t. $e^2 = \frac{1}{2} p_1(M)$. Then there exists orientable $\xi \subseteq TM$ of Cartan type with $e(\xi) = e$.

Proof.

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\begin{array}{c}
\Gamma(Gr_2(TM) \to M) \leftarrow \mathcal{C} \\
\Gamma(J^2 Gr_2(TM) \to M) \leftarrow \Gamma(R \to M)
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\]

\[
j^2 \downarrow \quad \downarrow j^2 \quad \simeq \text{according to Gromov's h-principle}
\]
Distributions of Cartan type on open 5-manifolds

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**Proof.**

\[
\Gamma(\text{Gr}_2(TM) \to M) \xleftarrow{\mathcal{C}} j^2 \downarrow \xrightarrow{\mathcal{C}} \Gamma(J^2 \text{Gr}_2(TM) \to M) \xleftarrow{\Gamma(R \to M)}
\]

\[j^2 \simeq \text{according to Gromov's h-principle}\]

\[
\mathcal{L}: R \to F/H, \quad H = \left\{ \begin{pmatrix} A & * & * \\ 0 & \det A & * \\ 0 & 0 & \det(A)A \end{pmatrix} : A \in \text{GL}_2(\mathbb{R}) \right\} \subseteq \text{GL}_5(\mathbb{R})
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Distributions of Cartan type on open 5-manifolds

**Theorem (Dave and H.)**

Suppose \( M \) open, spinnable 5-manifold, \( e \in H^2(M; \mathbb{Z}) \) s.t. \( e^2 = \frac{1}{2} p_1(M) \). Then there exists orientable \( \xi \subseteq TM \) of Cartan type with \( e(\xi) = e \).

**Proof.**

\[
\begin{align*}
\Gamma(\text{Gr}_2(TM) \to M) &\xleftarrow{\mathcal{C}} \\Gamma(J^2 \text{Gr}_2(TM) \to M) \\
j^2 &\xrightarrow{\text{contractible}} j^2 \\
\Gamma(J^2 \text{Gr}_2(TM) \to M) &\xleftarrow{\Gamma(\mathcal{R} \to M)} \\Gamma(\mathcal{R} \to M) \\
\mathcal{L}^* &\xrightarrow{} \Gamma(F/H \to M)
\end{align*}
\]

\( \mathcal{L} : \mathcal{R} \to F/H, \quad H = \left\{ \begin{pmatrix} A & * \\ 0 & \det A \end{pmatrix} : A \in \text{GL}_2(\mathbb{R}) \right\} \subseteq \text{GL}_5(\mathbb{R}) \)
Distributions of Cartan type on open 5-manifolds

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Suppose $M$ open, spinnable 5-manifold, $e \in H^2(M; \mathbb{Z})$ s.t. $e^2 = \frac{1}{2} p_1(M)$. Then there exists orientable $\xi \subseteq TM$ of Cartan type with $e(\xi) = e$.

**Proof.**

\[
\begin{array}{ccc}
\Gamma(Gr_2(TM) \to M) & \leftarrow & \mathcal{C} \\
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\end{array}
\]

$j^2 \simeq$ according to Gromov's h-principle

\[
\begin{array}{ccc}
\Gamma(F/H \to M) & \leftarrow & \mathcal{L} \\
\downarrow \mathcal{L}_* & & \\
\end{array}
\]

$L_* \simeq$ since fibers of $L$ contractible

$L : R \to F/H$, $H = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} \begin{pmatrix} * & * \\ 0 & \det(A)A \end{pmatrix} : A \in \text{GL}_2(\mathbb{R}) \right\} \subseteq \text{GL}_5(\mathbb{R})$
Existence on closed 5-manifolds

Does the h-principle hold on closed 5-manifolds, i.e., is

\[ j^2 : C \to \Gamma(R \to M) \] a (weak) homotopy equivalence?

Does \((S^2 \times S^3) \# (S^2 \times S^3) \# (S^2 \times S^3)\) admit a Cartan distribution?

(smale's classification)

Does \(S^5 \{ \text{three points} \}\) admit a Cartan distribution which is asymptotically flat, i.e. all three ends are diffeomorphic to the end of the simply connected nilpotent Lie group \(N\).

Does the existence of a Cartan distribution impose restrictions on \(\frac{1}{2} \pi_1(M), \pi_1(M)\), or the Reidemeister torsion of \(M\)?

Which mapping tori admit such structures?

What kind of surgery can be performed? (analogue of Lutz twist?)

\(\Sigma \times N\) admits a Cartan distribution if \(\Sigma\) is a closed connected surface with \(\chi(\Sigma) \geq -1\) and \(N\) a closed orientable 3-manifold. [Dave-H.]

\(\Sigma \times T^3\) does not admit a principal \(T^3\)-connection whose 2-plane bundle is of Cartan type if \(\chi(\Sigma) \leq 0\). [Dave-H.]
Existence on closed 5-manifolds

Does the h-principle hold on closed 5-manifolds, i.e., is 
\[ j^2 : \mathcal{C} \rightarrow \Gamma(\mathcal{R} \rightarrow M) \] a (weak) homotopy equivalence?

- Does \((S^2 \times S^3) \# (S^2 \times S^3) \# (S^2 \times S^3)\) admit a Cartan distribution? (Smale’s classification)
- Does \(S^5 \setminus \{\text{three points}\}\) admit a Cartan distribution which is asymptotically flat, i.e. all three ends are diffeomorphic to the end of the simply connected nilpotent Lie group \(N\).
- Does the existence of a Cartan distribution impose restrictions on \(\frac{1}{2} p_1(M), \pi_1(M)\), or the Reidemeister torsion of \(M\)? Which mapping tori admit such structures?
- What kind of surgery can be performed? (analogue of Lutz twist?)
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Does the h-principle hold on closed 5-manifolds, i.e., is $j^2 : C \to \Gamma(\mathcal{R} \to M)$ a (weak) homotopy equivalence?

- Does $(S^2 \times S^3) \# (S^2 \times S^3) \# (S^2 \times S^3)$ admit a Cartan distribution? (Smale’s classification)
- Does $S^5 \setminus \{\text{three points}\}$ admit a Cartan distribution which is asymptotically flat, i.e. all three ends are diffeomorphic to the end of the simply connected nilpotent Lie group $N$.
- Does the existence of a Cartan distribution impose restrictions on $\frac{1}{2} p_1(M)$, $\pi_1(M)$, or the Reidemeister torsion of $M$?
  Which mapping tori admit such structures?
- What kind of surgery can be performed? (analogue of Lutz twist?)
- $\Sigma \times N$ admits a Cartan distribution if $\Sigma$ is a closed connected surface with $\chi(\Sigma) \geq -1$ and $N$ a closed orientable 3-manifold. [Dave-H.]
- $\Sigma \times T^3$ does not admit a principal $T^3$-connection whose 2-plane bundle is of Cartan type if $\chi(\Sigma) \leq 0$. [Dave-H.]
Rumin type complex (BGG sequences)

Natural sequence of differential operators [Čap–Slovák–Souček]

\[
\Gamma(\mathcal{H}_0) \xrightarrow{D_0} \Gamma(\mathcal{H}_1) \xrightarrow{D_1} \Gamma(\mathcal{H}_2) \xrightarrow{D_2} \Gamma(\mathcal{H}_3) \xrightarrow{D_3} \Gamma(\mathcal{H}_4) \xrightarrow{D_4} \Gamma(\mathcal{H}_5)
\]

- $\mathcal{H}_i \to M$ natural vector bundles of ranks 1, 2, 3, 3, 2, 1
- $D_i$ of Heisenberg order $r_i = 1, 3, 2, 3, 1$
- computes de Rham cohomology, $D_i \oplus D_i' = L \circ d_i \circ L^{-1}$
- Rumin complex not elliptic but hypoelliptic (Rockland complex)
Rumin type complex (BGG sequences)

Natural sequence of differential operators [Čap–Slovák–Souček]

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\Gamma(H_0) & \xrightarrow{D_0} \Gamma(H_1) \xrightarrow{D_1} \Gamma(H_2) \xrightarrow{D_2} \Gamma(H_3) \xrightarrow{D_3} \Gamma(H_4) \xrightarrow{D_4} \Gamma(H_5) \\
\end{align*} \]

- \( H_i \to M \) natural vector bundles of ranks 1, 2, 3, 3, 2, 1
- \( D_i \) of Heisenberg order \( r_i = 1, 3, 2, 3, 1 \)
- computes de Rham cohomology, \( D_i \oplus D'_i = L \circ d_i \circ L^{-1} \)
- Rumin complex not elliptic but hypoelliptic (Rockland complex)
- Every BGG sequence associated with irreducible \( G_2 \)-representation is Rockland [Dave-H]. In this case \( H_i = G_0 \times G_0 H_i(p_+; E) \).
- reduction of structure group to maximal compact \( K_0 \subseteq G_0 \) induces volume density on \( M \) and Hermitian metric on \( H_i \), providing standard \( L^2 \)-inner product on \( \Gamma(H_i) \), whence formal adjoints.
Theorem (Hypoellipticity)

If \( \psi \in \Gamma^{-\infty}(\mathcal{H}_i) \) such that \( D_i \psi \) and \( D^*_i \psi \) smooth, then \( \psi \) smooth.
Analytic results

Theorem (Hypoellipticity)

If \( \psi \in \Gamma^{-\infty}({\mathcal{H}}_i) \) such that \( D_i \psi \) and \( D_{i-1}^* \psi \) smooth, then \( \psi \) smooth.

Theorem (Maximal hypoelliptic estimate)

If \( M \) closed, then \( \ker(D_i) \cap \ker(D_{i-1}^*) \subseteq \Gamma^\infty({\mathcal{H}}_i) \) finite dimensional, and

\[
\| \psi \|_s \leq C \left( \| D_i \psi \|_{s-r_i} + \| D_{i-1}^* \psi \|_{s-r_{i-1}} + \| Q \psi \| \right),
\]

where \( Q \) denotes orthogonal projection onto \( \ker(D_i) \cap \ker(D_{i-1}^*) \).
Analytic results

**Theorem (Hypoellipticity)**

If $\psi \in \Gamma^{-\infty}(\mathcal{H}_i)$ such that $D_i \psi$ and $D_{i-1}^* \psi$ smooth, then $\psi$ smooth.

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If $M$ closed, then $\ker(D_i) \cap \ker(D_{i-1}^*) \subseteq \Gamma^\infty(\mathcal{H}_i)$ finite dimensional, and

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$$

where $Q$ denotes orthogonal projection onto $\ker(D_i) \cap \ker(D_{i-1}^*)$.

**Corollary (Hodge decomposition)**

If $M$ closed then each de Rham cohomology class admits unique (harmonic) representative in $\ker(D_i) \cap \ker(D_{i-1}^*)$. 
Key points in the analysis

- Rumin–Seshadri operator of Heisenberg order $\kappa$
  \[ \Delta := (D_i^* D_i)^{s_i} + (D_{i-1} D_{i-1}^*)^{s_i-1} \]
  
  where $\kappa = 2s_i r_i = 2s_{i-1} r_{i-1}$

- osculating group $\mathcal{T}_x M \cong N$, the simply connected nilpotent Lie group with Lie algebra $\text{gr}(\mathcal{T}_x M) \cong \mathfrak{n}$

- Heisenberg principal symbol $\sigma_x(\Delta)$, is a left invariant differential operator on $\mathcal{T}_x M$

- Rockland condition: $\pi(\sigma_x(\Delta))$ injective for each non-trivial irreducible unitary representation $\pi: \mathcal{T}_x M \to U(\mathcal{H})$

- Rockland theorem: $\sigma_x(\Delta)$ invertible

- Yuncken–van Erp calculus:
  \[ \Psi^s \text{ pseudodifferential operators of Heisenberg order } s \]
  \[ \text{parametrix } P \in \Psi^{-\kappa} \text{ such that } \Delta P - \text{id and } P \Delta - \text{id smoothing} \]

- Heisenberg–Sobolev scale: For each real $s$ exists $\Lambda \in \Psi^s$ and $\Lambda' \in \Psi^{-s}$ such that $\Lambda \Lambda' - \text{id and } \Lambda' \Lambda - \text{id smoothing operators.}$
Applications on closed manifolds

- Schwartz kernel $k_t$ of $e^{-t\Delta}$ is smooth and

$$k_t(x, x) \sim \sum_{j=0}^{\infty} q_j(x) t^{(j-n)/\kappa}, \quad n = 10.$$

- $\Delta$ essentially selfadjoint with compact resolvent. Weyl’s law:

$$\# \text{ eigenvalues less than } \lambda \sim \alpha \cdot \text{vol}(M) \lambda^{n/\kappa}$$
Applications on closed manifolds

- Schwartz kernel \( k_t \) of \( e^{-t\Delta} \) is smooth and

\[
k_t(x, x) \sim \sum_{j=0}^{\infty} q_j(x) t^{(j-n)/\kappa}, \quad n = 10.
\]

- \( \Delta \) essentially selfadjoint with compact resolvent. Weyl’s law:

\[
\# \text{ eigenvalues less than } \lambda \sim \alpha \cdot \text{vol}(\mathcal{M}) \lambda^{n/\kappa}
\]

- \( \zeta(s) = \text{tr}(\Delta^{-s}) \) holomorphic for \( \Re(s) > n/\kappa \), meromorphic on \( \mathbb{C} \), at most simple poles at \( s = (n - j)/\kappa \) where \( j \in \mathbb{N}_0 \), holomorphic at \( s \in \mathbb{N}_0 \)

- Rumin–Seshadri type analytic torsion, \( k_{i+1} - k_i = N_i \),

\[
\log \tau = -\frac{1}{\kappa} \sum_i (-1)^i N_i \zeta'_\Delta(0)
\]

- Dependence on reduction to \( K_0 \subset G_0 \)? Cheeger–Müller theorem?
Thank you for your attention!

Preprints (joint with Shantanu Dave):

- Graded hypoellipticity of BGG sequences.
  arXiv:1705.01659
- On 5-manifolds admitting rank two distributions of Cartan type.
  arXiv:1603.09700

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