Rank two distributions of Cartan type on 5-manifolds
(based on joint work with Shantanu Dave)

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Outline

1. A geometry in five dimensions
2. Topological aspects
3. The h-principle
4. Analysis of BGG sequences
5. Analytic torsion of Rumin complex
A geometry in five dimensions

Definition

A rank two distribution $\xi \subseteq TM$ is called of Cartan type if it is bracket generating with growth vector $(2,3,5)$.

- I.e., locally there exist sections $X, Y \in \Gamma^\infty(\xi)$ such that $X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]]$ is a frame of $TM$.
- A.k.a. generic rank two distributions in dimension five.
A geometry in five dimensions

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- I.e., locally there exist sections \( X, Y \in \Gamma^\infty(\xi) \) such that \( X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]] \) is a frame of \( TM \).
- A.k.a. generic rank two distributions in dimension five.
- These are filtered manifolds,

\[
TM = T^{-3}M \supseteq T^{-2}M \supseteq T^{-1}M \supseteq T^0M = 0,
\]

with osculating algebras \( \text{gr}(T_xM) \cong n = n_{-3} \oplus n_{-2} \oplus n_{-1} \) where

\[
[X, [X, Y]], [Y, [X, Y]], [X, Y], X, Y.
\]
A geometry in five dimensions (cont.)

Examples include:

- Lie group $N$ with graded nilpotent Lie algebra $\mathfrak{n}$
- $G_2/P \cong S^2 \times S^3$ (flat model, locally diffeomorphic to $N$)
- surface rolling on another surface (5-dim. configuration space) without slipping and twisting (encoded in rank two distribution) is of Cartan type iff Gaussian curvatures disjoint
A geometry in five dimensions (cont.)

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Regular **parabolic geometry** of type \((G_2, P)\):

- canonical (normal) Cartan connection
- symmetry group of dimension at most 14
- Cartan’s curvature tensor (a section of \( S^4 \xi^* \)) vanishes iff locally diffeomorphic to flat model \( G_2/P \)
- curved BGG sequences [Čap–Slovák–Souček]
- canonical conformal metric of signature \((2, 3)\) with conformal holonomy contained in \( G_2 \) [Nurowski]
Topological aspects

If $M$ admits orientable distribution of Cartan type $\xi \subseteq TM$, then $TM \cong \xi \oplus \varepsilon_1 \oplus \xi$. (1)

In particular, $M$ spinnable and $\frac{1}{2}p_1(M) = e(\xi)^2$.

If $M$ closed, then [Atiyah] $0 = k(M) = \sum_{q \text{ even}} \dim H^q(M; \mathbb{R}) \mod 2$.

Theorem (Dave and H.)

a) If $M$ open, spinnable, and $\frac{1}{2}p_1(M) = e(\xi)^2$, then (1) holds.

b) If $M$ closed, spinnable, $\frac{1}{2}p_1(M) = e(\xi)^2$, and $k(M) = 0$, then (1) holds.
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The h-principle

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Suppose $M$ is an open, spinnable 5-manifold, $e \in H_2(M; \mathbb{Z})$ such that $e^2 = \frac{1}{2} p_1(M)$. Then there exists an orientable $\xi \subseteq TM$ of Cartan type with $e(\xi) = e$.

Application of Gromov's h-principle:

$C \xrightarrow{j_2} \Gamma(R) \xrightarrow{L} \Gamma(F/H)$

$C \subseteq \Gamma(Gr_2(TM))$ space of all rank two distributions of Cartan type

$R \subseteq J_2 Gr_2(TM)$ open $\text{Diff}(M)$-invariant subbundle

$L: R \to F/H$ where $F$ denotes the frame bundle and $H = \{ (A^* \ast \ast 0 \det A^* \ast \ast 0 \det(A^*)) : A \in \text{GL}_2(R) \} \subseteq \text{GL}_5(R)$

Fundamental question for closed 5-manifolds:

To what extent is $j_2: C \to \Gamma(R)$ a (weak) homotopy equivalence?
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$C \xrightarrow{j_2} \Gamma(R) \xrightarrow{L} \cong \Gamma(F/H)$

where $R$ is an open Diff($M$)-invariant subbundle of $\Gamma(Gr_2(TM))$, $F$ denotes the frame bundle, and $H$ is the subgroup of $GL_2(R)$ defined by:

$$H = \{ (A \ast 0 0 \ast 0 0) \det(A) \ast A \in GL_2(R) \} \subseteq GL_5(R).$$

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\begin{align*}
\mathcal{C} & \xrightarrow{j^2} \Gamma(\mathcal{R}) \xrightarrow{L^*} \Gamma(F/H) \\
\mathcal{C} & \subseteq \Gamma(\text{Gr}_2(TM)) \text{ space of all rank two distributions of Cartan type} \\
\mathcal{R} & \subseteq J^2 \text{Gr}_2(TM) \text{ open Diff}(M)\text{-invariant subbundle} \\
L : \mathcal{R} & \rightarrow F/H \text{ where } F \text{ denotes the frame bundle and } \\
H & = \left\{ \begin{pmatrix} A & * \\ 0 & \det A \end{pmatrix} \right\} : A \in \text{GL}_2(\mathbb{R}) \subseteq \text{GL}_5(\mathbb{R})
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The \textbf{h-principle}

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- $C \subseteq \Gamma(\text{Gr}_2(TM))$ space of all rank two distributions of Cartan type
- $\mathcal{R} \subseteq J^2 \text{Gr}_2(TM)$ open $\text{Diff}(M)$-invariant subbundle
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\textbf{Fundamental question for closed 5-manifolds:}

To what extent is $j^2 : C \rightarrow \Gamma(\mathcal{R})$ a (weak) homotopy equivalence?
The h-principle (cont.)

Which **closed** 5-manifolds admit rank 2 distribution of Cartan type?

- Does \((S^2 \times S^3) \# (S^2 \times S^3) \# (S^2 \times S^3)\) admit a Cartan distribution? (Smale’s classification)
- Does \(S^5 \setminus \{\text{three points}\}\) admit a Cartan distribution such that all three ends are diffeomorphic to the end of \(N\).
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- Does the existence of a Cartan distribution impose restrictions on \(\frac{1}{2} p_1(M)\), \(\pi_1(M)\), or the Reidemeister torsion of \(M\)?
- Which mapping tori or circle bundles admit such structures?
- What kind of surgery can be performed? (analogue of Lutz twist?)

Theorem (Dave and H.)

a) \(\Sigma \times N\) admits a Cartan distribution if \(\Sigma\) is a closed connected surface with \(\chi(\Sigma) \geq -1\) and \(N\) a closed orientable 3-manifold.

b) \(\Sigma \times T^3\) does not admit a principal \(T^3\)-connection whose 2-plane bundle is of Cartan type if \(\chi(\Sigma) \leq 0\).
The h-principle (cont.)

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BGG sequences

Natural sequence of differential operators [Čap–Slovák–Souček]

\[ \Gamma(E^0) \xrightarrow{D_0} \Gamma(E^1) \xrightarrow{D_1} \Gamma(E^2) \xrightarrow{D_2} \Gamma(E^3) \xrightarrow{D_3} \Gamma(E^4) \xrightarrow{D_4} \Gamma(E^5) \]
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constructed using associated Cartan geometry of type \((G_2, P)\)

- \(\mathcal{G} \to M\) principal \(P\)-bundle, \(P \subseteq G_2\) parabolic
- normal Cartan connection \(\omega \in \Omega^1(\mathcal{G}; g_2)\)
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- normal Cartan connection \(\omega \in \Omega^1(G; \mathfrak{g}_2)\)

and irreducible \(G_2\)-representation \(E\) with heighest weight \(a\lambda_1 + b\lambda_2\)

- \(E^k = G \times_P H^k(n; E)\) natural vector bundles of ranks
  \(a + 1, a + b + 2, 2a + b + 3, 2a + b + 3, a + b + 2, a + 1\)
- \(D_k\) natural differential operators of Heisenberg orders \(r_k = b + 1, 3a + 3, 2b + 2, 3a + 3, b + 1\)
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Linear connection \(d^\nabla_k : \Omega^k(M; G \times_P E) \to \Omega^{k+1}(M; G \times_P E)\)

induced by Cartan connection, \(D_k = \pi \circ d^\nabla_k \circ L\)
Rockland sequences

BGG sequences are Rockland, i.e., Heisenberg principal symbol sequence

\[ \cdots \to C^\infty(\mathcal{T}_x M) \otimes \mathcal{E}_x^k \xrightarrow{\sigma_x(D_k)} C^\infty(\mathcal{T}_x M) \otimes \mathcal{E}_x^{k+1} \to \cdots \]

becomes exact in every non-trivial irreducible unitary representation \( \pi \) of the osculating group \( \mathcal{T}_x M \cong N \) (non-commutative tangent space),

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Hence, Rumin–Seshadri operator

\[ \Delta = \Delta_k := (D_k^* D_k)^{s_k} + (D_{k-1}^* D_{k-1}^*)^{s_{k-1}} \]

is Rockland of Heisenberg order \( \kappa = 2s_k r_k = 2s_{k-1} r_{k-1} \).

Formal adjoints w.r. to standard \( L^2 \)-inner products
The Rockland theorem

**Theorem (Dave and H.)**

There exists parametrix $A \in \Psi^{-\kappa}$ such that $A\Delta - \text{id}$ is a smoothing operator. In particular, $\Delta$ is hypoelliptic.
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- \( \Psi^s \) ... PDO of Heisenberg order \( s \in \mathbb{C} \) [van Erp and Yuncken]
- special instance of a more general Rockland theorem for Rockland PDOs on general filtered manifolds, generalizing results of Helffer–Nourrigat, Folland–Stein, Beals–Greiner, Ponge, Christ–Geller–Głowacki–Polin, ...
- uses Heisenberg PDO calculus of Erp and Yuncken, analysis on nilpotent Lie groups by Christ–Geller–Głowacki–Polin, and arguments of Ponge.
Corollary (Maximal hypoelliptic estimate)

On closed manifolds \( \ker(\Delta) \) is finite dimensional and

\[
\| \psi \|_{H^s} \leq C \left( \| \Delta \psi \|_{H^{s-\kappa}} + \| P \psi \|_{\ker(\Delta)} \right)
\]

where \( P \) denotes the orthogonal projection onto \( \ker(\Delta) \).

Heisenberg Sobolev scale \( H^s \), \( s \in \mathbb{R} \), well adapted to Heisenberg PDOs.
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Heisenberg Sobolev scale $H^s$, $s \in \mathbb{R}$, well adapted to Heisenberg PDOs.

Corollary

$\Delta$ essentially selfadjoint with compact resolvent and domain of closure $H^\kappa$. In particular, the spectrum of $\Delta$ is discrete.
The heat equation on closed manifolds

Obtain strongly differentiable semigroup $e^{-t\Delta}$, $t > 0$.

$k_t(x, y)$ ... distributional kernel of $e^{-t\Delta}$, $t > 0$. 

Theorem (Dave and H.) $e^{-t\Delta}$ is a smoothing operator, and $k_t(x, y)$ depends smoothly on $t$.

Moreover, for $t \downarrow 0$ we have an asymptotic expansion $k_t(x, x) \sim \infty \sum_{j=0}^{\infty} t^{(j-10)}/\kappa q_j(x)$, where $q_j \in \Gamma^\infty(\text{end}(E) \otimes |\Lambda^1 M|)$, $q_j = 0$ for all odd $j$, and $q_0(x) > 0$.

Proof: $\Delta + \partial / \partial t$ on $M \times \mathbb{R}$. $q_j(x)$ in principle locally computable (parametrix of $\Delta + \partial / \partial t$).
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- $q_j(x)$ in principle locally computable (parametrix of $\Delta + \frac{\partial}{\partial t}$)
The heat equation (cont.)

Corollary (Heat trace asymptotics)

For $t \searrow 0$ we have asymptotic expansion

$$\text{tr}(e^{-t\Delta}) \sim \sum_{j=0}^{\infty} t^{(j-10)/\kappa} a_j$$

where $a_j = \int_M \text{tr}(q_j)$, $a_j = 0$ for all odd $j$, and $a_0 > 0$. 

Corollary (Weyl's law)

There exists complete ONS of eigenvectors $\psi_j \in \Gamma_\infty(E)$, $\Delta \psi_j = \lambda_j \psi_j$, $\lambda_j \geq 0$.

There exists universal constant $\alpha$ (depending only on $G_2$-rep. and $k$) s.t.

$$\# \{j \in \mathbb{N} : \lambda_j \leq \lambda\} \sim \alpha \cdot \text{vol}(M) \lambda^{10/\kappa}$$
as $\lambda \to \infty$ for a natural class of $L^2$ inner products associated with fiberwise Euclidean metric on 2-plane bundle $\xi \subseteq TM$. 

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Rank two distributions on 5-manifolds

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The heat equation (cont.)

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for a natural class of $L^2$ inner products associated with fiberwise Euclidean metric on 2-plane bundle $\xi \subseteq TM$. 
Complex powers on closed manifolds

**Theorem (Dave and H.)**

For closed $M$ we have $\Delta^{-z} \in \Psi^{-\kappa z}$ for all $z \in \mathbb{C}$.

Moreover, the zeta function

$$\zeta(z) = \text{tr}(\Delta^{-z}) = \int_M \text{tr}(k_{\Delta^{-z}}(x,x)) \, dx, \quad \Re(z) > 10/\kappa,$$

admits meromorphic continuation to entire complex plane with at most simple poles located at $(10 - j)/\kappa$, $j \in \mathbb{N}_0$, but holomorphic at $z \in -\mathbb{N}_0$. 
Hodge decomposition for Rumin complex

For the trivial $G_2$-representation the BGG sequence

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\Gamma(\mathcal{E}^0) \xrightarrow{D_0} \Gamma(\mathcal{E}^1) \xrightarrow{D_1} \Gamma(\mathcal{E}^2) \xrightarrow{D_2} \Gamma(\mathcal{E}^3) \xrightarrow{D_3} \Gamma(\mathcal{E}^4) \xrightarrow{D_4} \Gamma(\mathcal{E}^5)
$$

is a Rumin complex computing de Rham cohomology,

$$D \oplus B = L^{-1} \circ d \circ L.$$

In this case, ranks of vector bundles $\mathcal{E}^k$ are 1, 2, 3, 3, 2, 1, and Heisenberg orders of $D_k$ are $r_k = 1, 3, 2, 3, 1$. 

**Corollary (Hodge decomposition)**

For closed $\mathcal{M}$ we have orthogonal decomposition

$$
\Gamma^\infty(\mathcal{E}^k) = D_k^{-1}(\Gamma^\infty(\mathcal{E}^{k-1})) \oplus \ker(\Delta^k) \cap \ker(D^k) \oplus D^{\ast}k(\Gamma^\infty(\mathcal{E}^{k+1})).
$$

In particular, each de Rham cohomology class admits unique (harmonic) representative in $\ker(\Delta^k) = \ker(D_k) \cap \ker(D^{\ast}k-1)$. 


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In particular, each de Rham cohomology class admits unique (harmonic) representative in $\ker(\Delta_k) = \ker(D_k) \cap \ker(D^*_{k-1})$. 

Analytic torsion of the Rumin complex

- Twisting Rumin complex with flat vector bundle $F$.
- Fix numbers $N_k$ such that $N_{k+1} - N_k = r_k$, e.g. $N_k = 0, 1, 4, 6, 9, 10$. 

\[
\text{Definition (Analytic torsion)}
\parallel - \parallel \text{sdet}(H^\ast(M; F))_{\text{an}} := \exp(-1^k \sum_{k} (-1)^k N_k \zeta' \Delta_k(0)) \cdot \parallel - \parallel \text{sdet}(H^\ast(M; F))
\]

\[
\text{Hodge decomposition to get metric on graded determinant line}
\text{sdet}(H^\ast(M; F)) := \bigotimes_{k} \text{det}(H^k(M; F))(-1)^k = \bigotimes_{k} \text{det}(\text{ker}(\Delta_k))(-1)^k
\]

This analytic torsion does not depend on choice of $s_k$ or $N_k$.
Twisting Rumin complex with flat vector bundle $F$.

Fix numbers $N_k$ such that $N_{k+1} - N_k = r_k$, e.g. $N_k = 0, 1, 4, 6, 9, 10$.

**Definition (Analytic torsion)**

$$\| - \|_{\text{an}}^{\text{sdet}(H^*(M;F))} := \exp \left( -\frac{1}{\kappa} \sum_k (-1)^k N_k \zeta'_{\Delta_k}(0) \right) \cdot \| - \|_{\text{sdet}(H^*(M;F))}^{\text{Hodge}}$$
Analytic torsion of the Rumin complex

- Twisting Rumin complex with flat vector bundle $F$.
- Fix numbers $N_k$ such that $N_{k+1} - N_k = r_k$, e.g. $N_k = 0, 1, 4, 6, 9, 10$.

**Definition (Analytic torsion)**

\[
\| - \|_{\text{an}}^{\text{sdet}(H^*(M;F))} := \exp \left( - \frac{1}{\kappa} \sum_k (-1)^k N_k \zeta'_{\Delta_k}(0) \right) \cdot \| - \|_{\text{Hodge}}^{\text{sdet}(H^*(M;F))}
\]

- Use Hodge decomposition to get metric on graded determinant line

\[
\text{sdet}(H^*(M;F)) := \bigotimes_k \det(H^k(M;F))^{(-1)^k} = \bigotimes_k \det(\ker(\Delta_k))^{(-1)^k}
\]

This analytic torsion does not depend on choice of $s_k$ or $N_k$. 
Variation of the $L^2$-inner product

- $\langle \phi, \psi \rangle_{E_k, u} = \langle G_{k, u} \phi, \psi \rangle_{E_k, u_0}$ where $\phi, \psi \in \Gamma^\infty(E_k)$, $u \in \mathbb{R}$
- $G_{k, u}$ vector bundle automorphism of $E_k$
- $\dot{G}_{k, u} = G_{k, u}^{-1} \frac{\partial}{\partial u} G_{k, u}$ infinitesimal variation
Variation of the $L^2$-inner product

- $\langle \phi, \psi \rangle_{\mathcal{E}_k,u} = \langle \mathcal{G}_{k,u} \phi, \psi \rangle_{\mathcal{E}_k,u_0}$ where $\phi, \psi \in \Gamma^\infty(\mathcal{E}_k), \ u \in \mathbb{R}$
- $\mathcal{G}_{k,u}$ vector bundle automorphism of $\mathcal{E}_k$
- $\dot{\mathcal{G}}_{k,u} = \mathcal{G}_{k,u}^{-1} \frac{\partial}{\partial u} \mathcal{G}_{k,u}$ infinitesimal variation

Theorem (Anomaly formula)

$$\frac{\partial}{\partial u} \log \| - \|_{\text{an},u}^{\text{sdet}(H^*(M;F))} = \frac{1}{2} \int_M \sum_k (-1)^k \text{tr}(\dot{\mathcal{G}}_{k,u} q_{10}^{\Delta_{k,u}})$$

Goal: comparison with Reidemeister torsion (Cheeger–Müller theorem)
Variation of the $L^2$-inner product

- $\langle \langle \phi, \psi \rangle \rangle_{E_k, u} = \langle \langle G_{k,u} \phi, \psi \rangle \rangle_{E_k, u_0}$ where $\phi, \psi \in \Gamma^\infty(\mathcal{E}_k)$, $u \in \mathbb{R}$
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**Theorem (Anomaly formula)**

$$\frac{\partial}{\partial u} \log \| - \|_{\text{an},u}^{\text{sdet}(H^*(M;F))} = \frac{1}{2} \int_M \sum_k (-1)^k \text{tr}(\dot{G}_{k,u} q_{10}^\Delta_{k,u})$$

Here $q_{10}^\Delta_{k,u} \in \Gamma^\infty(\text{end}(\mathcal{E}_k) \otimes |\Lambda_M|)$ constant term in expansion

$$k_t^\Delta_{k,u} (x, x) \sim \sum_{j=0}^{\infty} t^{(j-10)/\kappa} q_j^\Delta_{k,u} (x)$$

locally computable (in principle)
Variation of the $L^2$-inner product

- $\langle\langle \phi, \psi \rangle \rangle_{E_k,u} = \langle\langle G_{k,u} \phi, \psi \rangle \rangle_{E_k,u_0}$ where $\phi, \psi \in \Gamma^\infty(E_k), \ u \in \mathbb{R}$
- $G_{k,u}$ vector bundle automorphism of $E_k$
- $\dot{G}_{k,u} = G_{k,u}^{-1} \frac{\partial}{\partial u} G_{k,u}$ infinitesimal variation

**Theorem (Anomaly formula)**

$$\frac{\partial}{\partial u} \log \| - \|^\text{sdet}(H^\ast(M;F))_{an,u} = \frac{1}{2} \int_M \sum_k (-1)^k \text{tr}(\dot{G}_{k,u} q_{10}^{\Delta_{k,u}})$$

Here $q_{10}^{\Delta_{k,u}} \in \Gamma^\infty(\text{end}(E_k) \otimes |\Lambda M|)$ constant term in expansion

$$k_t^{\Delta_{k,u}}(x, x) \sim \sum_{j=0}^\infty t^{(j-10)/\kappa} q_j^{\Delta_{k,u}}(x)$$

locally computable (in principle)

**Goal:** comparison with Reidemeister torsion (Cheeger–Müller theorem)
Thank you for your attention!

Preprints (joint with Shantanu Dave):

- On 5-manifolds admitting rank two distributions of Cartan type.  
  Trans. Amer. Math. Soc. (accepted)  
  arXiv:1603.09700

- Graded hypoellipticity of BGG sequences.  
  arXiv:1705.01659

- The heat asymptotics on filtered manifolds.  
  arXiv:1712.07104

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