The amazing power of dimensional analysis: Scaling laws in finance

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Mathias Pohl,
joint work with Alexander Ristig, Ludovic Tangpi and Walter Schachermayer.
Dimensional analysis

Market impact

Trading activity
  Empirical results

Time scaling of volatility
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Scaling laws in finance
**Dimensional analysis: Pendulum**

Assume that the period $p$ of a pendulum (with $[p] = \mathbb{T}$) is fully explained by

- the rod’s length $l$, with $[l] = \mathbb{L}$,
- the bob’s mass $m$, with $[m] = \mathbb{M}$,
- the acceleration $g$ of the gravitation, with $[g] = \mathbb{L}/\mathbb{T}^2$.

Hence, we assume

$$p = f(l, m, g).$$
Dimensional analysis: Pendulum

Ansatz:

\[ p = f(l, m, g) = c \cdot l^{y_1} m^{y_2} g^{y_3}. \]

Solving the following linear system

\[
\begin{array}{ccc|c}
 l & m & g & p \\
 \hline
 L & 1 & 0 & 1 & 0 \\
 M & 0 & 1 & 0 & 0 \\
 T & 0 & 0 & -2 & 1 \\
\end{array}
\]

\[
y_1 + y_3 = 0 \\
y_2 = 0 \\
-2y_3 = 1
\]

yields the solution

\[ p = c \cdot \sqrt{\frac{l}{g}}. \]
Dimensional analysis: Pendulum

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\[ y_1 + y_3 = 0 \]
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yields the solution

\[ p = c \cdot \sqrt{\frac{l}{g}}. \]

The ansatz does not restrict the generality of the relation!
Dimensional analysis: Pendulum

What if the period $p$ of a pendulum also depends on the amplitude $a$?

$$p = f(l, m, g, a).$$

- Assume $[a] = \mathbb{L}$.
- Then, the resulting linear system does not have a unique solution anymore.

- It follows that,

$$p = \sqrt{\frac{l}{g}} h \left( \frac{a}{l} \right).$$

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Market impact

The **market impact** of a meta-order is the size of the price movement caused by the execution of the meta-order.

- Bouchaud et al. (2009): non-linear (square root) and fading market impact.

  - **Empirical support:** Bacro et al. (2015); Bershova and Rakhlin (2013); Brokmann et al. (2015); Engle et al. (2012); Gomes and Waelbroeck (2015); Mastromatteo et al. (2014); Moro et al. (2009); Toth et al. (2011).
  - **Contradicting evidence:** Almgren et al. (2005); Zarinelli et al. (2015); Capponi and Cont (2019).
  - **Theoretical justification:** Donier et al. (2015); Kyle and Obizhaeva (2017).
Market impact

The **market impact** of a meta-order is the size of the price movement caused by the execution of the meta-order.

- Denote the market impact by $G$.
- Measured it in percentage of the price.
- *Square-root law for market impact:* The market impact $G$ is proportional to the square root of the size $Q$ of the meta-order, i.e., $G \sim \sqrt{Q}$.
Market impact

We identify the variables which are expected to influence the market impact $G$:

- $Q$ the size of the meta-order,

- $P$ the price of the stock,

- $V$ the traded volume of the stock,

- $\sigma^2$ the squared volatility of the stock,
Market impact

We identify the variables which are expected to influence the market impact $G$:

- $Q$ the size of the meta-order, measured in units of shares $[Q] = S$,
- $P$ the price of the stock, measured in units of money per share $[P] = U/S$,
- $V$ the traded volume of the stock, measured in units of shares per time $[V] = S/T$,
- $\sigma^2$ the squared volatility of the stock, we assume $[\sigma^2] = T^{-1}$. 
Market impact

Assumption 1
The market impact $G$ depends only on the above 4 variables, i.e.,

$$G = g(Q, P, V, \sigma^2),$$

where $g : \mathbb{R}^4_+ \to \mathbb{R}_+$ and $G$ are invariant under changes of the units chosen to measure the “dimensions” $S, T, U$. 

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Market impact

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Under Assumption 1, the market impact is of the form

$$G = f \left( \frac{Q \sigma^2}{V} \right),$$

where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an arbitrary function.
Market impact

Inspired by Modigliani and Miller (1958) and Kyle and Obizhaeva (2017), we impose the following.

Assumption 2 (Leverage neutrality)

The market value of any firm is independent of its capital structure.
Market impact

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**Assumption 2 (Leverage neutrality)**
The market value of any firm is independent of its capital structure.

Assume a share of a stocks yields a dividend of $P/2$. Then,

\[
P \rightarrow P/2, \quad \sigma \rightarrow 2\sigma, \\
V \rightarrow V, \quad G \rightarrow 2G, \\
Q \rightarrow Q.
\]
Market impact

Inspired by Modigliani and Miller (1958) and Kyle and Obizhaeva (2017), we impose the following.

Assumption 2 (Leverage neutrality)
Scaling the Modigliani-Miller “dimension” $\mathbb{M}$ by a factor $A \in \mathbb{R}_+$ implies that

$$P \rightarrow A^{-1}P,$$
$$\sigma^2 \rightarrow A^2 \sigma^2,$$
$$V \rightarrow V,$$
$$G \rightarrow AG,$$
$$Q \rightarrow Q.$$
Market impact

Theorem (Square-root law for market impact)

Under Assumption 1, i.e., \( G = g(Q, P, V, \sigma^2) \), and Assumption 2, i.e., leverage neutrality, then there is a constant \( c > 0 \) such that

\[
G = c \cdot \sigma \sqrt{\frac{Q}{V}}.
\]
Market impact

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Proof:

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( P )</th>
<th>( V )</th>
<th>( \sigma^2 )</th>
<th>( G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( U )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( T )</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( M )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Scaling laws in finance
Market impact

Including additional explanatory variables implies more general “laws”:

- $T$ the length of the execution interval, measured in units of time $[T] = \mathbb{T}$.
  
  $G = \sigma \sqrt{\frac{Q}{V}} h \left( \frac{Q}{VT} \right)$, for some function $h$.

- $C$ the “bet cost”, measured in units of money $[C] = \mathbb{U}$.
  
  $G = \left( \frac{PV}{\sigma^2 C} \right)^{1/3} f \left( \left( \frac{Q^3 P^2 \sigma^2}{VC^2} \right)^{1/3} \right)$, for some function $f$. 

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Trading activity

- Trading activity := traded volume is proportional to price variability:
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¬ Trading activity := number of trades is proportional to price variability:
  ▶ Jones et al. (1994); Ané and Geman (2000); Dufour and Engle (2000).
Trading activity

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- Trading activity := number of trades is proportional to price variability:
  - Jones et al. (1994); Ané and Geman (2000); Dufour and Engle (2000).

- Trading activity := number of trades is proportional to
  \[ ((\text{price}) \times (\text{volatility}) \times (\text{traded volume}))^{2/3} \].
  - Madhavan et al. (1997); Wyart et al. (2008); Kyle and Obizhaeva (2016), Andersen et al. (2016); Benzaquen et al. (2016); Bucci et al. (2019).
We are interested in explaining the arrival rate of trades in a given stock measured as

\[ N \] the number of trades within a fixed time interval \([t, t + T]\), measured per units of time \([N] = T^{-1}\).
Trading activity

We identify the variables which are expected to influence $N$:

- $V$ the traded volume of the stock in the interval $[t, t + T]$, measured in units of shares per time $V = \frac{S}{T}$,

- $P$ the average price of the stock in the interval $[t, t + T]$, measured in units of money per share $P = \frac{U}{S}$,

- $\sigma^2$ the variance of the log-returns in the interval $[t, t + T]$,
Trading activity

We identify the variables which are expected to influence \( N \):

- \( V \) the traded volume of the stock in the interval \([t, t + T]\), measured in units of shares per time \([V] = \text{S/\text{T}}\),
- \( P \) the average price of the stock in the interval \([t, t + T]\), measured in units of money per share \([P] = \text{U/\text{S}}\),
- \( \sigma^2 \) the variance of the log-price in the interval \([t, t + T]\), we assume \([\sigma^2] = \text{T}^{-1}\).
Trading activity

Proposition
Assume that \( N \) depends only on the 3 variables \( \sigma^2, P \) and \( V \), i.e.,

\[
N = g(\sigma^2, P, V),
\]

where \( g : \mathbb{R}^3_+ \to \mathbb{R}_+ \) and \( G \) are invariant under changes of the units chosen to measure the “dimensions” \( S, T, U \). Then, there is a constant \( c > 0 \) such that

\[
N = c \cdot \sigma^2.
\]
Trading activity

Proposition
Assume that $N$ depends only on the 3 variables $\sigma^2$, $P$ and $V$, i.e.,

$$N = g(\sigma^2, P, V),$$

where $g : \mathbb{R}_+^3 \to \mathbb{R}_+$ and $G$ are invariant under changes of the units chosen to measure the “dimensions” $S$, $T$, $U$. Then, there is a constant $c > 0$ such that

$$N = c \cdot \sigma^2.$$

This relation was investigated e.g. in Jones et al. (1994), but it is too simplistic!

Scaling laws in finance
Trading activity

We identify further variables which are expected to influence \( N \):

- \( S \) the average bid-ask spread in the interval \([t, t + T]\), measured in units of money per share \([S] = \mathbb{U}/\mathbb{S}\),
- \( C \) the average cost per trade in the interval \([t, t + T]\), measured in units of money \([C] = \mathbb{U}\).

Let \( Q = N/V \) denote the average order size, then \( C = QS \).
Trading activity

Theorem (3/2-Law)
Assume that \( N \) depends only on the 4 variables \( \sigma^2, P, V \) and \( C \), i.e.,

\[
N = g(\sigma^2, P, V, C),
\]

where \( g : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+ \) and \( G \) are invariant under changes of the units chosen to measure the “dimensions” \( S, T, U \) and \( M \), i.e., the assumption of leverage neutrality holds. Then, there is a constant \( c > 0 \) such that

\[
N^{3/2} = c \cdot \frac{\sigma PV}{C}.
\]
Trading activity

- **3/2-Law:**

\[ N^{3/2} = c \cdot \frac{\sigma PV}{C}. \]

- Since \( C = SQ = SV/N \), the same relation can be written as

\[ \frac{S}{P} = c \cdot \frac{\sigma}{\sqrt{N}}. \]
Trading activity

▷ 3/2-Law:

\[ N^{3/2} = c \cdot \frac{\sigma PV}{C}. \]

▷ Since \( C = SQ = SV/N \), the same relation can be written as

\[ \frac{S}{P} = c \cdot \frac{\sigma}{\sqrt{N}}. \]

▷ Moreover, we find a connection to the Square Root Law:

\[ G \approx \frac{S}{P} = c \cdot \sigma \sqrt{\frac{Q}{V}}. \]
Empirical results

Using LOBSTER database (https://lobsterdata.com), we consider $d = 128$ sufficiently liquid stocks with high market capitalizations in period from 2.1.2015 to 31.8.2015.

Let us fix an interval length $T \in \{30, 60, 120, 180, 360\}$ min and an asset $i \in \{1, \ldots, d\}$ and let $j \in \{1, \ldots, n\}$ refer to an arbitrary interval. Suppose the trades in the considered interval $j$ arrive at irregularly spaced transaction times $t_1, t_2, \ldots, t_{N_j}$. Then,

- $N_j$ denotes the number of trades in the interval $j$,
- $Q_j = N_j^{-1} \sum_{k=1}^{N_j} Q_{t_k}$ denotes the average size of the trades in the interval $j$, where $Q_{t_k}$ denotes the number of shares traded at time $t_k$,
- $V_j = N_j \times Q_j$ is the traded volume in the interval $j$,
- $P_j = N_j^{-1} \sum_{k=1}^{N_j} P_{t_k}$ denotes the average midquote price in the interval $j$, where $P_{t_k} = (A_{t_k} + B_{t_k})/2$ and $A_{t_k}$ (resp. $B_{t_k}$) denotes the best ask (resp. bid) price after the transaction at time $t_k$,
- $\hat{\sigma}_j^2$ denotes the estimated squared volatility in the interval $j$,
- $S_j = N_j^{-1} \sum_{k=1}^{N_j} S_{t_k}$ denotes the average bid-ask spread in the interval $j$, where $S_{t_k} = A_{t_k} - B_{t_k}$ is the bid-ask spread after the transaction at time $t_k$, and
- $C_j = Q_j \times S_j$ is the cost per trade in the interval $j$. 

Scaling laws in finance
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Figure 1: We consider the fixed interval length $T = 60\text{min}$. The lines indicate the estimated linear relations between the considered quantities.
\[ N \sim \sigma^2 \quad \text{versus} \quad N \sim \left(\frac{\sigma PV}{C}\right)^{2/3} \]

Figure 2: We consider the fixed interval length \( T = 60\text{min} \). The lines indicate the estimated linear relations between the considered quantities, illustrating the empirical results.
$N \sim \sigma^2 \textit{ versus } N \sim \left(\frac{\sigma PV}{C}\right)^{2/3}$

Consider

$$N \sim (\sigma^2)^{1-\gamma} \left(\frac{PV}{C}\right)^\gamma.$$ 

Then,

- $\gamma = 0$ corresponds to $N \sim \sigma^2$.
- $\gamma = 2/3$ corresponds to $N \sim \left(\frac{\sigma PV}{C}\right)^{2/3}$.

A logarithmic transformation yields the linear model

$$\log(N_{ij}) = \alpha_i + \beta_i \log(\hat{\sigma}_{ij}^2) + \gamma_i \log\left(\frac{P_{ij}V_{ij}}{C_{ij}}\right) + \varepsilon_{ij},$$

with the restriction $\beta_i + \gamma_i = 1$ for fixed $T$ and fixed asset $i$. 

Scaling laws in finance
\[ N \sim \sigma^2 \quad \text{versus} \quad N \sim \left( \frac{\sigma PV}{C} \right)^{2/3} \]

Figure 3: For different assets \( i \) and different interval lengths \( T \), we estimate \( \hat{\gamma}_i \) by ordinary least squares. The panels show kernel density estimates across the estimated parameters \( \hat{\gamma}_i \) for \( T \in \{30, 120, 360\} \) min.
On the universality of the 3/2-law

To check whether there is empirical support for

\[ N^{3/2} = c \cdot \frac{\sigma PV}{C}, \]

we compute for a fixed interval length \( T \) and a fixed asset \( i \)

\[
\hat{c}_i = n^{-1} \sum_{j=1}^{n} \frac{C_{ij} N_{ij}^{3/2}}{\hat{\sigma}_{ij} P_{ij} V_{ij}} = n^{-1} \sum_{j=1}^{n} \frac{N_{ij}^{1/2}}{\hat{\sigma}_{ij}} \frac{S_{ij}}{P_{ij}}, \quad \text{for} \quad i = 1, \ldots, d.
\]
On the universality of the $3/2$-law

Figure 4: The left panel shows the computed values for $\hat{c}_i$ in dependence of $T \in \{30, 60, 120, 180, 360\}$ min. The right panel shows a kernel density estimate across the estimates $\hat{c}_i$ for fixed $T = 120$ min.
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Time scaling of volatility

- Instead of the classical Black-Scholes model, we consider
  \[ P_t = P_0 \exp(\sigma W_t^H). \]
- A consistent estimator for the parameter \( \sigma \) is
  \[ \hat{\sigma}(H) := \left( \sum_{k=2}^{N} | \log(P_{t_k}) - \log(P_{t_{k-1}}) |^{1/H} \right)^H. \]
- We obtain a more general time scaling of the volatility
  \[ [\hat{\sigma}(H)] = T^{-H}. \]
- The 3/2-Law turns into a (1 + \( H \))-Law:
  \[ N^{1+H} = c \cdot \frac{\hat{\sigma}(H)PV}{C}. \]
Time scaling of volatility

Can we determine an $H \in (0, 1)$ that minimizes the cross-sectional dispersion across of $c$ in the $(1 + H)$-Law?

- We compute

$$\hat{c}_i(H) = n^{-1} \sum_{j=1}^{n} \frac{N_{ij}^{1+H} C_{ij}}{\hat{\sigma}_{ij}(H) P_{ij} V_{ij}},$$

for all stocks $i$ in our sample.

- We find empirically that overall the constant $\hat{c}_i(H)$ typically increases in $H$.

- Hence, we consider the Gini-coefficient $G$ as a scale invariant measure for the variation in $\hat{c}_i(H)$.
Figure 5: The left panel illustrates the Gini-coefficient in dependence of $H$ for different $T \in \{30, 60, 120, 180, 360\}$. The right panel shows the computed values for $\hat{c}_i(\hat{H})$ such that $\hat{H}$ minimizes the Gini-coefficient for fixed $T \in \{30, 60, 120, 180, 360\}$. 
References

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Market Microstructure and Liquidity 3, 3&4.

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The amazing power of dimensional analysis: quantifying market impact.
Market Microstructure and Liquidity 3, 3&4.

Pohl, M., Ristig, A., Schachermayer, W., Tangpi, L. (2018)
Theoretical and empirical analysis of trading activity
Mathematical Programming.
Thank you for your attention!

Figure 6: Bodensee on June 31, 2019.
Latent order book

Figure 7: The V-shaped latent order book is the most basic argument to explain the Square Root Law for market impact.